Abstract

Doubly fractal integer sequences are introduced and study by C. Kimberling. The connection of these sequences and the signature sequences is questioned and worked by him and some other mathematicians. It is proved that every signature sequence is a doubly fractal. In this article we discuss the converse relation and prove the equivalency of these two family of sequences.

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1 Introduction

If each positive integer is infinitely many times repeated in a sequence

2 Definitions and preliminaries

Suppose $S = (s_1, s_2, s_3, \cdots)$ is a sequence of positive integers which may be finite or infinite. For the finite case we may use S_N to emphasize the size of S. The kth term of a sequence S is denoted by $S(k)$.

Definition 1. Let $S = (s_1, s_2, s_3, \cdots)$ be an integer sequence. The trimmed sequence obtained by removing the first occurrence of every integer number from S is denoted by $\wedge_S = S_U$ and is called the upper-trimmed(sub)sequence of S.

Definition 2. The sequence $S = (s_1, s_2, \cdots)$ of positive integers is lower-trimmed by subtracting 1 from every term and deleting zero terms. We denote the obtained sequence by $\vee_S = S_L$.

If an infinite sequence S with positive integer terms has three properties below, then it will be a doubly fractal sequence:

$$(i) S(1) = 1;$$
$$(ii) \wedge_S = S;$$
$$(iii) \vee_S = S;$$

If S is a finite sequence, the equality in properties (ii) and (iii) is checked by the initial segment of S.

Definition 3. Let $S = (s_1, s_2, \cdots)$ be a sequence of positive integers. For every integer $n \in (N)$, the index of kth occurrence of n in S is denoted by $I_k(n)$.

Lemma 4. *The $n+1$ term of initial segment of every doubly fractal sequence S is either in the form of $(1, 2, 3, \cdots n, 1)$ or $(1, 1, 1, \cdots, 1, 2)$, for $n \geq 2$.*

Proof. In the former case, by the property (iii), for every $1 \leq k \leq n$, $I_1(k)$ in S should equal to $I_1(k)$ in $\vee?_S$. Since the first 1 term is deleted in lower-trimmed sequence, the position of each term is shifted left, but simultaneously the value of them is reduced once. So the values and positions of remaining terms become the same as the original sequence. On the other hand, if k is the biggest integer which has not occurred in initial segment of S, $1 < k < n$, then $k \in \vee_S$. The latter case shows the exceptional case in which the second term is chosen 1. □

The sequence below is an example of a doubly fractal sequence which the second term is 1.

$$S_e = (1, 1, 1, 2, 1, 2, 1, 2, 3, 1, 2, 3, 1, 2, \cdots)$$

2.1 Signature sequences

Let θ be a real number (rational or irrational) and let $M_\theta = \{i + j\theta | i, j \in (N)\}$ be a multi-set depend on θ. We order the elements of M as usual order of real numbers, and in the case of equal terms, we order the equal elements with respect to i's. The result ordered set is called the signature (sequence) of θ. For example, the signature sequence of $\theta = \sqrt{2}$ is $1, 2, 1, 3, 2, 1, 4, 3, 2, 5 \cdots$ and the signature of $1/7$ is $1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 2, 2, 1, 2, 1, 2, 1, 2, \cdots$.

Lemma 5. *For a positive real number θ, the signature sequence of θ is doubly fractal sequence.*

Proof. If S is a signature sequence, for any $i, j \in (N)$ if $i + j\theta < h + k\theta \leq i + (j + 1)\theta$ and we upper-trim the sequence then again the result is the same as S because $i + (j - 1)\theta < h + (k - 1)\theta < i + j\theta$. The relation for (iii) will be $i + j\theta < h + k\theta < i + 1 + j\theta$ and the proof is similar. □

3 The Construction

We want to explain how a doubly fractal sequence can be made. By lemma 1, the initial segment of a doubly fractal S is either $(1, 2, 3, \cdots, n, 1)$ or$(1, 1, 1, \cdots, 1, 2)$. We assume the former case, and the latter can be translated to the former. The terms of this initial segment is referred by *main terms*. Start with $S = (1, 2, 3, \cdots, n, 1)$. We extend this sequence so that the result is always a doubly fractal (finite) sequence. Every subsequence from each 1 the next number n after that is called a *block*. The least positive integer which has not appeared in S so far is denoted by ℓ_S. To extend S, we rewrite the main terms after the initial block, and insert ℓ_S's between each two main terms step by step, i.e., we have

$$S = (1, 2, 3, \cdots, n, 1, n + 1, 2, n + 2, 3, n + 3, \cdots, n - 1, 2n - 1, n)$$

For example if $n = 4$ we have $A = (1, 2, 3, 4, 1, 5, 2, 6, 3, 7, 4)$. The next ℓ_S is $2n$. Until now the sequence S is extended two *block*. For kth level extension,

$k \geq 3$, we proceed as follows. Consider the sequence t obtained by adding 1 to all terms between $n-1$ and n, exclusive, in the last block, and the sequence t' consist of the terms between last $n+1$ and its previous term n, exclusive (e.g., in sequence A we have $t = (8)$ and $t' = (1)$).

Proposition 6. *The ℓ_S and 1 appear once in t and t', respectively, and all of other terms of t and t' are the same.*

Proof. Generally, for any $k > 1$, if terms between $I_k(n)$ and $I_k(n+1)$ are lower-trimmed, they must turn to terms between $I_k(n-1)$ and $I_k(n)$ and if they are upper-trimmed they must turn to terms between $I_(k-1)(n)$ and $I_(k-1)(n+1)$. If terms between $I_2(n)$ and $I_2(n+1)$ are upper-trimmed, they must turn to subsequence (1) because there's only one 1 between $I_1(n)$ and $I_1(n+1)$. Upper-trimming a (sub)sequence will delete its ell_S's and the only term 1 that is ell_S is the first term of the sequence. Therefore 1 only occurs once between each $I_k(n)$ and $I_k(n+1)$. The proof for t is similar. If terms between $I_(k-1)(n)$ and $I_k(n+1)$ are lower-trimmed, they must turn to terms between $I_(k-1)(n-1)$ and $I_(k-1)(n)$. If terms between $I_k(n-1)$ and $I_k(n)$ are upper-trimmed they must turn to the terms between $I_(k-1)(n-1)$ and $I_(k-1)(n)$. It is known that there's one 1 between $I_(k-1)(n)$ and $I_k(n)$ and one ℓ_S between $I_k(n-1)$ and $I_k(n)$. *means that if t is upper-trimmed (which means the ℓ_S in t is removed) and t' is lower-trimmed (the 1 in t' is removed) the result will be the same.(terms between $I_(k-1)(n-1)$ and $I_(k-1)(n)$.* □

We denote by $Indx_t(\ell_S)$ and $Indx_{t'}(1)$ the position of ℓ_S and 1 in sequences t and t' respectively. We construct a new sequence P from t and t' by merging them under the following conditions. Write all terms of t' into the P except 1. If $Indx_t(\ell_S) = Indx_{t'}(1)$ then insert one of the subsequences $(1, \ell_S)$ or $(\ell_S, 1)$ in the position $Indx_t(\ell_S)$ into the P sequence. For example, in the case of $A = (1, 2, 3, 4, 1, 5, 2, 6, 3, 7, 4)$, the next ℓ_S will be 8 and P would be () or (8). On the other hand, if $Indx_t(\ell_S) \neq Indx_{t'}(1)$ then insert the ℓ_S and 1 into the P in the positions $Indx_t(\ell_S)$ and $Indx_{t'}(1)$, respectively, "from the right". For example, if $t = (a_1, \ell_S, a_2, a_3, a_4)$ and $t' = a_1, a_2, a_3, 1, a_4)$ then $P = (a_1, \ell_S, a_2, a_3, 1, a_4)$. Set $d = Indx_P(1) - Indx_P(\ell_S)$. Delete the term 1 and the terms after 1 in P (P may become a null sequence). To continue the sequence S, append P to the end of S, and append the last block of S after P. We want to insert new ℓ_S's into this new rewritten block. Start from beginning of this block, for each main term m, write corresponding ℓ_S in the position $Indx_S(\ell_S) = Indx_S(m) + d$ if and only if $I_k(1) < Indx_S(\ell_S) < I_k(n)$. In other words, if the index of the ℓ_S is out of the new block, we do not write it into the S.

$$A = (1, 2, 3, 4$$
$$1, 5, 2, 6, 3, 7, 4$$
$$1, 8, 5, 2, 9, 6, 3, 10, 7, 4).$$

We extend the sequence for two blocks to show an example which $Indx_t\ell_S \neq Indx_{t'}1$. In this block $t = (11, 8), t' = (1, 8)$. So if we $(11, 1)$ is placed in P and

the n1 and terms after 1 is removed, $P = (11)$, ℓ_S's would occur before the correspondent main term: $A = (1, 2, 3, 4, 1, 5, 2, 6, 3, 7, 4, 1, 8, 5, 2, 9, 6, 3, 10, 7, 4, 11, 1, 8, 5, 12, 2, 9, 6, 13, 3, 10, 7, 14, 4)$ Now $t = (11, 8, 15), t' = (11, 1, 8)$. At first $P = t$, therefore $Indx_t\ell_S \neq Indx_{t'}1$. Here $P = (11)$. Now $\ell_S = 15$ is still between 8 and 5 like t and 1 is still between 11 and 8 like t'. Since $d = 2$ all ℓ_S's will be two terms after their correspondent main term.

$$\begin{aligned} A = (1, 2, 3, 4, \\ 1, 5, 2, 6, 3, 7, 4, \\ 1, 8, 5, 2, 9, 6, 3, 10, 7, 4, 11, \\ 1, 8, 5, 12, 2, 9, 6, 13, 3, 10, 7, 14, 4, 11, \\ 1, 8, 15, 5, 12, 2, 9, 16, 6, 13, 3, 10, 17, 7, 14, 4). \end{aligned}$$

Theorem 7. *Every doubly fractal sequence S can be made by the construction procedure explained above.*

Proof. The conclusion will be obtained by induction. By Lemma 1, the initial block of every doubly fractal sequence is either $(1, 2, 3, ..., n, 1)$ or $(1, 1, 1, \cdots, 1, 2)$. This is coincide with the initial assumption in our construction. Now suppose that until the ith occurrence of term n in the sequence S is obtained by i extension steps of above procedure. Consider the terms between the ith and the $i + 1$th n in S. This subsequence of S is called $(i + 1)$*-th part* of S. Since S is doubly fractal, $\wedge_S$ should similar to S, so all main terms must be appeared in $i + 1$-th part. Having upper-trimmed property also conclude that the elements between the main terms and the P subsequence contained in ith part of S must be repeated in $(i + 1)$-th part of S with the same order. It also concludes that ℓ_S's are the only terms which haven't occurred in the ith part, but could occur in the $i + 1$th part. 1 is between terms n and $n + 1$ in the ith part, 1 must be between n and $n + 1$ in the $i + 1$th part of S. If the sequence is lower-trimmed, the terms between ith n and ith $n+1$ must turn to terms between $i - 1$th $n - 1$ and ith n, these terms are t explained in **??**. Therefore, terms between n and $n+1$ in the ith part are terms in t plus 1. There is one ℓ_S between $n - 1$ and n in ith part, that means there is one ℓ_S between n and $n + 1$ in the $i + 1$th part. It is explained that terms between t and t' must have the same order when they are repeated in terms between n and $n + 1$. By **??**, there's one ℓ_S in t and one 1 in t' and all other terms of t and t' have the same order. Therefore, if the 1 between n and $n + 1$ is deleted, the terms between n and $n + 1$ will be exactly t. If $Indx_t\ell_S = Indx_{t'}1$ both $(\ell_S, 1)$ and $(1, \ell_S)$ can be placed in the $i + 1$th part, because in both ways term have the same order which they had in t and t'. Otherwise, for example if $t = (t_1, t_2, \ell_S, t_3), t' = (1, t'_1, t'_2, t'_3$, which by **??** $t_1 = t'_1, t_2 = t'_2, t_3 = t'_3$; ℓ_S must be between t_2 and t_3 and 1 must be before t_1. Now before removing term 1 and terms after 1 in P, $P = (1, t_1, t_2, \ell_S, t_3)$. Now sth must hold on, because if the sequence is lower-trimmed 2 will turn to 1 and ℓ_S correspondent to 2 will turn to ℓ_S correspondent to 1. Since there's no 1 between 2 and its correspondent ℓ_S, the terms between these two terms must also turn to terms between 1 and its correspondent ℓ_S. Therefore sth holds on.

□

Theorem 8. *The constructed sequence by this algorithm is a signature sequence.*

Proof. By **??**, the first $n+1$ terms of each doubly fractal sequence is the initial segment of infinitely many signature sequences. P was the only part in the algorithm which its terms could be chosen. Since signature sequences are doubly fractal, the first term that could be in a doubly fractal sequence that is not a signature must be in P. The only choice was between $(k, 1, \ell_S)$ and $(k, \ell_S, 1)$. For the first case, $k + hx \leq \ell_S + x \leq 1 + h'x$. Since $\ell_S + x$ and $1 + h'x$ cannot be in the form of any inequality by adding or reducing some number. Because $k+hx$ is and terms before it are initial segment of a signature sequence, $k+hx$ is more than or equal to terms before it and terms after it are all less than it. □

4 Extension

Proposition 9. *Each positive θ has a unique signature sequence.*

Proof. We prove this by contradiction. We assume that $\alpha < \beta$ are two distinct numbers with the same signature sequences $\alpha = a.a_1a_2a_3\cdots a_m,\cdots$, $\beta = b.b_1.b_2.b_3\cdots b_m\cdots$, in which $a_k = b_k$, for all $k \leq m$, and $a_(m+1) \neq b_(m+1)$. Now for every $e, f, g, h \in \mathbb{N}$ we have $e + f\alpha < g + h\alpha$ if and only if $e + f\beta < g + h\beta$. Equivalently $0 < g - e + (h - f)\alpha$ if and only if $g - e + (h - f)\beta$. So we can write $0 < (-b_1b_2b_3\cdots b_mb_(m+1)) + \beta 10^(m+1)$ but we have $(-b_1b_2b_3\cdots b_mb_(m+1)) + \alpha 10^(m+1) < 0$, a contradiction. □

Proposition 10. *The $n+2$ first terms of a doubly fractal sequence can be the initial segment of infinitely many signature sequences.*

Proof. First $n+2$ terms of any doubly fractal sequence is $(1, 2, \cdots, n, 1, n+1)$. For some θ, $1+\theta < 2+\theta < \cdots < n+\theta \leq 1+2\theta \leq n+1+\theta$. So $n-1 < \theta \leq n$. This means that for any $n-1 < \theta \leq n$, the initial segment of the signature sequence is $(1, 2, \cdots, n, 1, n+1)$. Since each two different θ's have different signature sequences, there are infinitely many signature sequences that $n-1 < \theta \leq n$. □

The construction introduced produces every infinite doubly fractal sequence. Since all signature sequences are doubly fractal, all of them are constructed in this way. Every signature sequence is doubly fractal, the first $n+2$ terms of any doubly fractal sequence is the initial segment of infinitely many signature sequences.